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COMMENT

# Phase factor for charge–monopole system

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**Abstract.** We calculate the Green function for a charge-monopole system and show that a non-integrable phase factor emerges without using the classical notion of path.

Twelve years ago a formulation of gauge fields in terms of non-integrable phase factors was advocated (Wu and Yang 1975). Many of the arguments presented therein were based on the classical notion of 'path'. In this comment we show that for a charge-monopole system a quantum derivation of this phase factor can be given. This calculation lends us confidence that the phase factor really describes the transition amplitude for a charge moving about a monopole and is not a flux about some hypothetical path. Classically such paths make sense, but in quantum theory the concept of a path gives way to that of a transition amplitude.

We begin by writing down the Schrödinger equation for a charge $e$ of mass $M$ in the field of an infinitely massive monopole

$$(2M)^{-1}[\boldsymbol{p}-(e/c)\boldsymbol{A}]^2\psi = E\psi \tag{1}$$

where the vector potential $\boldsymbol{A}$ is given in spherical coordinates by

$$\boldsymbol{A} = \hat{\phi}(g/r)\tan(\theta/2) \tag{2}$$

with

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} = g\boldsymbol{r}/r^3.$$

The vector potential is singular along the negative $z$ axis. By imposing the Dirac quantisation condition

$$eg/\hbar c = s = \text{integer or half-integer} \tag{3}$$

it can be shown that henceforth this string singularity plays no role in the physics of the system (Dirac 1931). The eigensolutions of (1) have been known for some time now (Goldhaber 1965, Boulware *et al* 1976).

$$\begin{aligned}
&\psi_{Jm} = (2J+1)^{1/2} j_q(kr) d^J_{s-m\,s}(\theta)\,\mathrm{e}^{\mathrm{i}m\phi}\\
&J = |s|, |s|+1, |s|+2, \ldots\\
&m = -J, -J+1, \ldots, J-1, J\\
&k = (2ME)^{1/2} \qquad E>0\\
&q+\tfrac{1}{2} = [(J+\tfrac{1}{2})^2 - s^2]^{1/2}
\end{aligned} \tag{4}$$

where $j_q$ is a spherical Bessel function, $d^J$ is a rotation function and $s$ is defined by (3). There are no negative energy solutions of (1).

The Green function may now be written as an expansion in terms of the eigenfunctions given above

$$G(\boldsymbol{r}t;\boldsymbol{r}'t')=\sum_{Jm}(2J+1)\int\frac{d^3k}{(2\pi)^3}j_q(kr)j_q(kr')\exp\left(-i\frac{k^2}{2M}(t-t')\right)\times d^J_{s-m\,s}(\theta)d^J_{s-m\,s}(\theta')\exp[im(\phi-\phi')]. \tag{5}$$

Here we assume $t>t'$. The sum over $m$ may be carried out (Edmonds 1960)

$$\sum_m d^J_{s-m\,s}(\theta)d^J_{s-m\,s}(\theta')\exp[im(\phi-\phi')]=\sum D^J_{s\,s-m}(\phi,-\theta,-\phi)D^J_{s-m\,s}(\phi',\theta',-\phi')=D^J_{ss}(\alpha,\gamma,\beta) \tag{6}$$

where $(\alpha,\gamma,\beta)$ are the Euler angles corresponding to the successive rotations $(\phi',\theta',-\phi')$ and $(\phi,-\theta,-\phi)$

$$\cos\gamma'=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi'-\phi).$$

The $k$ integral is not convergent. However, if we go over to Euclidean space $(t-t')\to -i(t-t')$ the integral may be evaluated using Weber's formula (Watson 1958). In Minkowski space then our Green function takes the form ($J_\mu$ is a Bessel function)

$$G(\boldsymbol{r}t;\boldsymbol{r}'t')=\frac{M}{iT(rr')^{1/2}}\exp\left(\frac{iM}{2T}(r^2+r'^2)\right)e^{is\Omega}\times\sum_J\frac{2J+1}{4\pi}J_\mu\left(\frac{Mrr'}{T}\right)d^J_{ss}(\gamma)\,e^{-i\mu\pi/2} \tag{7}$$

where $\mu=q+\frac{1}{2}$, $T=t-t'$ and $\Omega=\alpha+\beta$ is the solid angle subtended at the origin by the $z$ axis and the vectors $\boldsymbol{r}$ and $\boldsymbol{r}'$. We see the phase factor $\exp(is\Omega)$ in (7). Note that our discussion includes the quantum mechanical interaction between the electric and magnetic charges. Several consequences may now be derived.

We may calculate the scattering amplitude as follows. The scattered wave $\psi^+_k(r)$ has the form

$$\psi^+_k(r)=\lim_{t'\to-\infty}\left(\frac{2\pi i|t'|}{M}\right)^{3/2}G(\boldsymbol{r},0;\boldsymbol{r}',t')\exp[i(\boldsymbol{k}\cdot\boldsymbol{r}-k^2t/2M)]$$

(Pechukas 1969). Substituting the expression we obtained above we have

$$\psi^+_k(r)\sim\frac{1}{2kr}e^{is\Omega}\sum_J(2J+1)d^J_{ss}(\gamma)(e^{ikr}e^{-i\pi\mu}+i\,e^{-ikr}). \tag{8}$$

The second term in (8) represents an incoming wave so that the scattered wave is simply

$$\psi_{\text{scat}}(r)\sim r^{-1}e^{is\Omega}f(\gamma)\,e^{ikr} \tag{9}$$

where the scattering amplitude $f(\gamma)$ is given by

$$f(\gamma)=\frac{1}{2k}\sum_{J=s}(2J+1)d^J_{ss}(\gamma)\,e^{-i\pi\mu}. \tag{10}$$

This expression is given in Boulware *et al* (1976).

An interpretation for (7) can be arrived at by considering the Green function for infinitesimal times. Replacing $T$ in (7) by $\varepsilon$ we write it as ($s\delta\Omega$ is a infinitesimal flux)

$$G(\boldsymbol{r}t;\, \boldsymbol{r}'t') = \frac{M}{\mathrm{i}\varepsilon(rr')^{1/2}} \exp\left(\mathrm{i}\frac{M}{2\varepsilon}(r^2+r'^2) + \mathrm{i}s\delta\Omega\right) \times \sum_J \frac{2J+1}{4\pi} I_\mu\left(\frac{Mrr'}{\mathrm{i}\varepsilon}\right) d^J_{ss}(\gamma). \tag{11}$$

$I_\mu$ is an associated Bessel function. The limit $\varepsilon \to 0$ may be obtained with the aid of the asymptotic expansion

$$I_a(x) \underset{x\to\infty}{\sim} \frac{1}{(2\pi x)^{1/2}} \exp\left(x - (a^2 - \tfrac{1}{4})\frac{1}{2x}\right). \tag{12}$$

The result is

$$G(\boldsymbol{r}t;\, \boldsymbol{r}'t') \underset{\varepsilon\to 0}{\sim} \left(\frac{M}{2\pi\mathrm{i}\varepsilon}\right)^{1/2} \frac{1}{rr'} \exp\left(\mathrm{i}\frac{M}{2\varepsilon}(\boldsymbol{r}-\boldsymbol{r}')^2 + \mathrm{i}s\delta\Omega + \mathrm{i}\frac{\varepsilon s^2}{2Mrr'}\right) \times \sum_J \frac{2J+1}{4} d^J_{ss}(\gamma) \exp\left(-\mathrm{i}J(J+1)\frac{\varepsilon}{2Mrr'}\right). \tag{13}$$

The sum over $J$ may be evaluated by rewriting a formula given in appendix A of Boulware *et al* (1976)

$$\sum_J (2J+1) i_J(-\mathrm{i}kr) d^J_{ss}(\theta) = \tfrac{1}{2}(\pi k\xi)^{1/2} \mathrm{e}^{\mathrm{i}k\eta/2}[\mathrm{e}^{-\mathrm{i}\pi/4} I_{s-\frac{1}{2}}(-\tfrac{1}{2}\mathrm{i}k\xi) - \mathrm{i} I_{s+\frac{1}{2}}(-\tfrac{1}{2}\mathrm{i}k\xi)]$$

where $\xi = r(1+\cos\theta)$, $\eta = r(1-\cos\theta)$. Applying expansion (12) on both sides of this result we obtain the desired expression

$$\sum_J (2J+1) d^J_{ss}(\gamma) \exp\left(-\frac{\mathrm{i}}{2kr}J(J+1)\right) \underset{k\to 0}{\sim} -2\mathrm{i}kr \exp(\mathrm{i}k\eta - \mathrm{i}s^2/k\xi).$$

The Green function for infinitesimal times is then

$$G(\boldsymbol{r}t;\, \boldsymbol{r}'t') \underset{\varepsilon\to 0}{\sim} \left(\frac{M}{2\pi\mathrm{i}\varepsilon}\right)^{3/2} \exp\left(\frac{\mathrm{i}M}{2\varepsilon}(\boldsymbol{r}-\boldsymbol{r}')^2 + \mathrm{i}s\delta\Omega + \mathrm{O}(\varepsilon^{3/2})\right). \tag{14}$$

Now it is known that the infinitesimal propagator in a vector field $\boldsymbol{A}$ is given by (Schulman 1981)

$$\left(\frac{M}{2\pi\mathrm{i}\varepsilon}\right)^{3/2} \exp\left(\frac{\mathrm{i}M}{2\varepsilon}(\boldsymbol{r}-\boldsymbol{r}')^2 + \mathrm{i}e(\boldsymbol{r}-\boldsymbol{r}')\cdot\boldsymbol{A}(\tfrac{1}{2}\boldsymbol{r}+\tfrac{1}{2}\boldsymbol{r}')\right). \tag{15}$$

Comparing the last two results we conclude that the flux term $s\delta\Omega$ in (14) comes from the line integral of $\boldsymbol{A}$. This line integral may be interpreted as an infinitesimal loop integral by completing the loop with circular arcs to the $z$ axis. Since these arcs intersect $\boldsymbol{A}$ at right angles they do not contribute to the loop integral. Moreover for the finite-time Green function the line integral of $\boldsymbol{A}$ must be responsible for the flux term $s\Omega$ in (7), although it is clear that it also contributes to the other factors in the Green function. Nevertheless the line integral of $\boldsymbol{A}$ cannot be looked upon as a simple loop integral since the quantum propagator involves a sum over paths. Thus the non-integrable flux factor hides a 'sum-over-loop integral' of the vector field. Note that the phase factor for an infinitesimal path is just a flux; for a finite path the amplitudes sum up in a complicated way although a flux still emerges.

We show that $s = $ half-integer (3). This is the Dirac quantisation condition. Consider the amplitude for the charge to make a complete circuit about the string: $(r, \theta, \phi) \rightarrow (r', \theta', \phi + 2\pi)$. According to the above discussion the Green function is proportional to $e^{is\Omega}$ where $\Omega$ is the solid angle subtended by a circular cap around the $z$ axis with polar angle $\theta$. Since the description of the system cannot depend on the position of the string we expect the Green function to remain unchanged if the string were rotated onto the positive $z$ direction. In this case the Green function would be proportional to $e^{-is\Omega'}$ where $\Omega'$ is the solid angle of the same cap as viewed from the negative $z$ direction. The minus sign is due to the direction of motion of the charge. We have then

$$e^{is\Omega} = e^{-is\Omega'}$$

or

$$e^{i4\pi s} = 1.$$

It follows that

$$4\pi s = 2\pi n \qquad n = 0, 1, 2, \ldots \tag{16}$$

which is the Dirac result (Dirac 1931). Our discussion does not make use of classical paths around the string.

The result just obtained allows us to speak of gauge transformation. A shift of the string from the negative to the positive $z$ direction is really a gauge transformation. Thus the Dirac condition can be looked upon as a statement of the invariance of the phase factor around a loop under a gauge transformation. The special choice of the $z$ axis for the string does not prevent us from selecting other directions. Thus the invariance of the phase factor is true for any gauge transformation provided the Dirac condition is satisfied. We may now follow up this discussion with that of Wu and Yang (1975) but this time without having to use the idea of a classical path about the monopole.

Presumably a similar calculation may be done for other systems as well.

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